# Complementarity, Necessity and Preferences

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*Reducing a necessity to its lower limit causes the MRS to change rapidly creating inelastic demand and complementarity. The analytic concept that defines necessity also underlies income and substitution effects allowing a reinterpretation of the Slutsky decomposition and providing a utility foundation for marginal revenue. For three or more goods, a good is essential if adding it to the system decreases the elasticity of substitution.*

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What links complements, necessities and the Slutsky decomposition? The answer has been hiding in plain sight in Hicks and Allen (1934) under the misleading title “coefficient of income variation”. We prefer to call it the preference elasticity because it is the elasticity of the marginal rate of substitution. This simple elasticity underlies both income and substitution affects and allows an obvious simplification of the Slutsky decomposition.

We cannot say why Hicks and Allen (1934) do not fully explore this elasticity but we can say why we are drawn to it. We believe the best measure of whether a good is essential or not is whether the indifference curves are defined when there is no consumption of the good. As it happens, this definition hinges on whether preference elasticities are greater than or equal to one. Therefore this one concept, not only underlies income effects, substitution effects and the Slutsky decomposition, it also identifies essential goods. This background allows us to solve one of the more enduring puzzles in economics. Why is it that A may be a gross substitute for B and B may be a gross complement of A? Our answer is that A is inessential and B essential. Now the asymmetry is natural. Finally, preference elasticities allow us to write down elasticities of demand for popular utility functions in a snap and provide a utility foundation for marginal revenue.

Some of our results are foreshadowed in the recent work of Chambers, Echinique and Shmaya (2010, 2011) who link complements and substitutes to the elasticity of the MRS. They do not, however, mention necessities or simplify the Slutsky decomposition. Fisher (1972) provides the links between demand elasticities and gross substitutes. We came to all these works rather late in the process of writing the paper and we owe a greater intellectual debt to Grandeville (1989) who links the Slutsky "diamond" to gross complements and gross substitutes. The primary source though is Hicks and Allen (1934) who would recognize all the steps but might be surprised in how they are put together. Frisch (1959) offers a more modern exposition that we will find quite useful.

This is a particularly opportune moment to take up the analysis given the current interest in the biological foundations of behavior. Robson (2001) provides an excellent survey that argues utility emerges early in the evolutionary process opening up the prospect that animal studies will prove important. Kagel, Battalio, Rachlin and Green (1981) provide just such experiments on animal consumers that focus on substitutes, complements and the essential or inessential nature of goods. Their findings, and ours, will match up nicely. This biological foundation underlies a conscious system of perceptions, attitudes and beliefs among humans that is explored by Kemp (1998). Therefore many complementary elements of utility analysis based on essential goods are readily at hand offering the promise that each strand of research will benefit from the other.

The paper first links necessity to preference elasticities. This is the most creative and original part of the paper. Once this is done, the links between necessity, preference elasticities, substitution effects and income effects are simple rearrangements of Hicks and Allen (1934). Section II takes up comparative statics, simplifies the Slutsky decomposition and provides an explanation for the asymmetry between a need and a want. Section III provides applications to marginal revenue, common utility functions and outlines the extension to three or more goods. Section IV discusses supporting research on animals and humans but also points out where additional work is needed. Section V concludes. An appendix analyses three goods.

## I. Modeling Necessity

We begin by exploring the fundamental nature of essential and inessential goods near the subsistence limit. The goal is to learn what we can from this unambiguous and unforgiving environment and then apply this knowledge to the inherent ambiguities of modern society. For example tap water and Evian are both capable of supporting life but Evian is marketed as a luxury good. Even tap water is used for green lawns and swimming pools so that its status as an essential in many current societies is suspect. These examples illustrate that economic progress has converted some goods from essentials to inessentials and the reverse is true as well. Computers, the internet and cell phones are all inessential for life but are essential to be a productive and connected member of society. We will tackle these issues later, for now we use an unambiguous and clear definition.

**DEFINITION 1:** *A good is a necessity if positive utility requires consumption of the good.*

Similarly, a good is inessential if positive utility is possible without consumption of the good. The definition has the great advantage of simplicity and leads to a test based on indifference curves. Given two goods, $x_1$ and $x_2$, if indifference curves are bound away from the $x_1$ axis or asymptotically approach the $x_1$ axis then $x_2$ is essential because some arbitrarily small quantity of $x_2$ is necessary for positive utility. The asymptotic case will absorb most of our time and attention. The critical fact is that the indifference curves must bend just enough to approach but not cut the axis. If they bend more slowly, then the curves cut the axis and the good is inessential.

We follow Hicks and Allen (1934) and analyze indifference curves in terms of the marginal rate of substitution so that all concepts are grounded in ordinal and not cardinal utility. Toward these ends, let utility be defined as $U = U(x_1, x_2)$ and let $u_i' \equiv (\partial U / \partial x_i)$. The marginal rate of substitution is the negative of the slope of an indifference curve: $f \equiv \frac{u_1'}{u_2'} = -\frac{dx_2}{dx_1}$. Figure 1 illustrates the indifference map for two essential goods where the indifference curves are bounded by the axes.

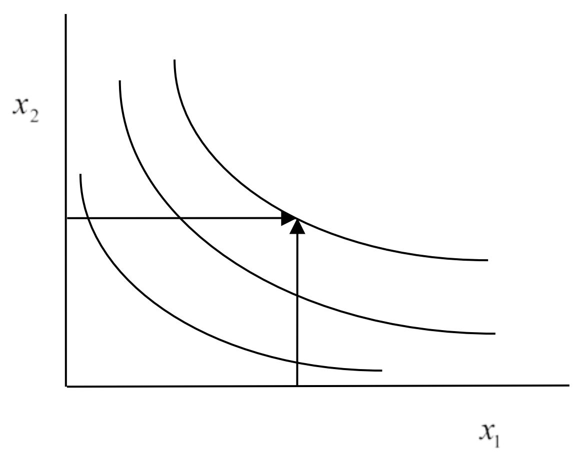


Figure 1. Two Essential Goods:
The MRS moderates as consumption rises.

The crucial feature is how indifference curves behave near their lower bound and we therefore begin by studying the behavior of the marginal rate of substitution as $x_i$ is increased or decreased. That is we depart from conventional practice and discuss how $f$ changes moving across indifference curves and not along a single indifference curve. Figure 1 illustrates for two necessities. Moving across indifference curves we expect $f$ to change from the extreme value of the relevant axis to something more moderate. Increasing $x_1$ the slope changes from nearly infinite to something moderate $\left(\partial f / \partial x_1 < 0\right)$ and increasing $x_2$ the slope changes from near zero to something moderate $\left(\partial f / \partial x_2 > 0\right)$. The change in the slope may be conveniently measured in elasticity form and the following sign conventions, borrowed from Hicks and Allen, assign positive elasticities to moderating influences on $f$. That is let

(1) $$\bar{\sigma}_1 \equiv -\frac{x_1}{f}\frac{\partial f}{\partial x_1}$$

and

(2) $$\bar{\sigma}_2 \equiv \frac{x_2}{f}\frac{\partial f}{\partial x_2}.$$

These preference elasticities are represented by $\bar{\sigma}_i$ because they are very closely related to elasticities of substitution traditionally represented by $\sigma$, as we show below. The overstruck bar is a reminder that one variable is held constant.

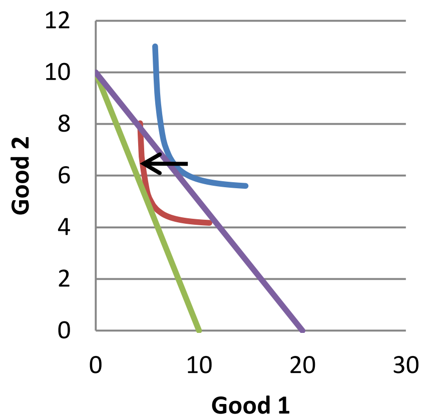


Figure 2a. A Need: Indifference slope changes rapidly

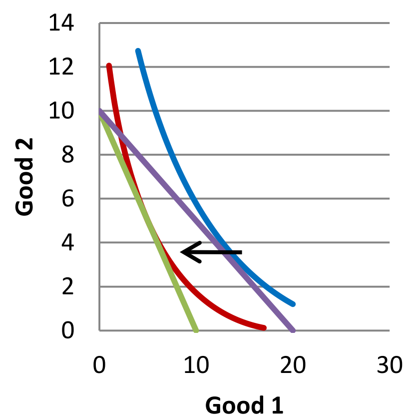


Figure 2b. A Want: Indifference slope changes slowly

Figure 2a shows how increasing the price of good 1 rotates the budget line and changes the equilibrium values of goods. Moving to the left along the black arrow the slope of the indifference curve changes more rapidly than the change in the slope of the budget line. This implies that the indifference curve becomes vertical before the budget line does and therefore the indifference curve never intersects the axis indicating the good is a necessity. This behavior also implies that demand for $x_2$ diminishes as $p_1$ rises and $x_2$ is a gross complement of $x_1$. In figure 2b, the indifference curves change slope relatively slowly and therefore do intersect the axes, the goods are wants and also substitutes.[1]

The division between rapid and slow changes in indifference curves is provided by $\bar{\sigma}_i = 1$ for $i = 1,2$. Given $\bar{\sigma}_1 = 1$ then $fx_1$ is a constant so that as $x_1 \to 0$ it must be true that $f \to \infty$ and the indifference curves never cut the axis.[2] Given

[1] At least two precursors to this idea exist. Chambers, Echinique and Shmaya (2010) connect gross complementarity to elastic preferences. However, the core concepts occurred to us while reading Grandville (1989). His figure 2 is particularly suggestive.

[2] Differentiate $fx_1 = k$ where $k$ is a constant and rearrange.

$\bar{\sigma}_1 > 1$ then $\lim_{x_1 \to 0} -\frac{1}{f}\frac{df}{dx_1} > \lim_{x_1 \to 0} \frac{1}{x_1} = \infty$ but nothing can be greater than infinity and we have reached a contradiction. The resolution is that $x_1$ cannot approach zero if $\bar{\sigma}_1 > 1$. The demonstration for $\bar{\sigma}_2 = 1$ is similar except that now $x_2 / f$ is a constant and as $x_2 \to 0$ it must be true that $f \to 0$ and once again the indifference curve becomes parallel to the axis as the axis is approached.

Of course a good is also essential if there is a minimum requirement as in the Stone-Geary utility function. These facts are summed up in the proposition below.

**PROPOSITION 1.** $x_i$ *is a necessity if either of the following hold:*

*(i)* $\bar{\sigma}_i \geq 1$ *as* $x_i \to 0$.

*(ii)* *A minimum subsistence requirement exists.*

For most of us, it is behavior away from the lower bound that is most relevant and we will say that a good is locally necessary if $\bar{\sigma}_i \geq 1$.

**DEFINITION 2:** $x_i$ *is locally necessary if* $\bar{\sigma}_i \geq 1$.

Whether water, Evian or the internet are locally necessary is determined by the elasticity of preferences. A small change in a necessity near the subsistence margin has a large impact on preference relations and we use this idea to generalize. If a small change in some quantity produces a disproportionate change in preferences then we say it is locally necessary.

*A. Necessity and the Elasticity of Substitution*

The elasticity of substitution is defined as

(3) $$\sigma = -\frac{d(x_2 / x_1)}{(x_2 / x_1)} \times \frac{f}{df}$$

Let $\chi = d\left(x_2 / x_1\right) / \left(x_2 / x_1\right)$ then expanding out $df$ we find that

(4) $$\sigma = \frac{\chi}{\left(\frac{f_1'}{f} dx_1 + \frac{f_2'}{f} dx_2\right)}.$$

Now rewrite the $dx_i$ in terms of $\chi$:

(5) $$\chi = \frac{x_1 dx_2 - x_2 dx_1}{x_1 x_2} = \left(\frac{x_1 dx_2 / dx_1 - x_2}{x_1 x_2}\right) dx_1 = \left(\frac{-y}{x_1 p_2 x_2}\right) = -\frac{dx_1}{x_1 s_2}.$$

The third equality follows from recognizing that in equilibrium $p_1 / p_2 = f = -dx_2 / dx_1$ and from the budget constraint $-x_1\left(p_1 / p_2\right) - x_2 = -y / p_2$ where $y$ is income. The last equality simply expresses the share of income spent on $x_2$, $p_2 x_2 / y$, as $s_2$. Rearranging (5) we find that $dx_1 = -\chi x_1 s_2$. Similarly, $dx_2 = \chi x_2 s_1$. In effect, the percentage change in the slope of the ray from the origin has been projected onto each axis. Substituting for the $dx_i$ in (4) allows us to express the elasticity of substitution in its final form[3]

(6) $$\sigma = \frac{1}{s_1 \bar{\sigma}_2 + s_2 \bar{\sigma}_1}.$$

From here it is apparent that the overall elasticity of substitution is a weighted average of the preference elasticities. If both goods are locally necessary then $\sigma \le 1$ while if both are inessential then $\sigma > 1$. This accords well with intuition: essential goods are poor substitutes.

[3] Hicks and Allen (1934 p. 201) derive a similar expression by a different method. In their equation $\kappa_i$ replaces $s_i$ and $\rho_j$ replaces $\bar{\sigma}_i$. That is their "coefficient of income variation" indexes the good held constant not the good that changes value and therefore their subscripts do not match ours.

*B. Necessity and the income effect*

Necessity is also linked to the income effect as the figure below illustrates. In the middle panel an increase in $x_2$ leaves indifference slope unaltered and rising income does not affect the demand for $x_1$. This illustrates that it is the value of $\bar{\sigma}_2$ that controls the income effect for $x_1$. In the left panel, $\bar{\sigma}_2 < 0$ and $x_1$ is inferior. In the right panel $\bar{\sigma}_2 > 0$ and $x_1$ is normal. The reason of course is that if returns to $x_2$ diminish, as indicated by $\bar{\sigma}_2 > 0$, then some part of income should shift to $x_1$.

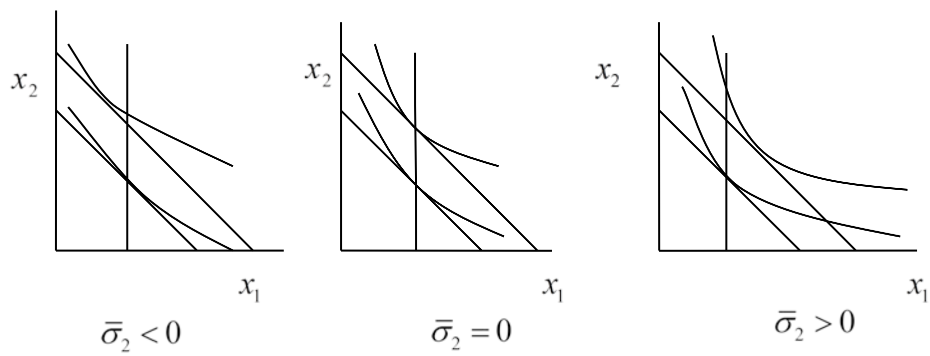


Figure 3: Preference Elasticities and the Income Effect

## II. Necessity and Comparative Statics

Our goal is to understand how these preference elasticities may be employed in comparative static analysis therefore we focus on the "normal" case where indifference curves are well-behaved, all income is spent and all consumption levels are strictly positive. Corner solutions and more general preference relations add complications that are well-understood. Therefore we assume the following general conditions throughout.

**General Conditions:** $U(x_1, x_2)$ is continuous and twice differentiable with $\partial U / \partial x_i > 0$ and $\partial^2 U / \partial x_i^2 < 0$ for $i = 1, 2$. The budget constraint is binding and utility is maximized at an interior solution with $0 < p_i < \infty$ and $x_i > 0$ for $i = 1, 2$ and $0 < \sigma < \infty$.

The first order conditions may be written as

(7) $$\frac{y}{p_2} = \frac{p_1}{p_2} x_1 + x_2$$

and

(8) $$f = \frac{p_1}{p_2}.$$

Totally differentiating the first order conditions generates the comparative static system

(9) $$\begin{bmatrix} f & 1 \\ f_1' & f_2' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \frac{1}{p_2} \begin{bmatrix} dy - x_1 dp_1 \\ dp_1 \end{bmatrix}.$$

The determinant of the system is closely related to the preference elasticities and the elasticity of substitution:

(10) $$\frac{p_2 x_1 x_2}{yf} \begin{vmatrix} f & 1 \\ f_1' & f_2' \end{vmatrix} = \frac{p_1 x_1}{y} \frac{x_2 f_2'}{f} - \frac{p_2 x_2}{y} \frac{x_1 f'}{f} = s_1 \bar{\sigma}_2 + s_2 \bar{\sigma}_1 = \frac{1}{\sigma}.$$

The income elasticity is

$$\eta_1 \equiv \frac{y}{x_1}\frac{dx_1}{dy} = \frac{y}{x_1}\frac{p_2 x_1 x_2}{yf}\frac{1}{p_2}\begin{vmatrix} 1 & 1 \\ 0 & f_2' \end{vmatrix} / \left(s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1\right) \tag{11}$$

$$\eta_1 = \frac{\bar{\sigma}_2}{s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1} = \bar{\sigma}_2\sigma.$$

Equation (11) contains quite a bit of information. First, it confirms the intuition from figure 3 that the income elasticity for $x_1$ depends on the sign of $\bar{\sigma}_2$. Second, it shows that homothetic utility functions $(\eta_i = 1)$ occur if $\bar{\sigma}_1 = \bar{\sigma}_2$. Third, if one good is essential and the other inessential then spending shifts to the inessential as income rises.[4]

*A. Reinterpreting the Slutsky Equation*

The elasticity of demand is

$$\varepsilon_1 \equiv -\frac{p_1}{x_1}\frac{dx_1}{dp_1} = \frac{-p_1}{x_1}\frac{p_2 x_1 x_2}{yf}\frac{1}{p_2}\begin{vmatrix} -x_1 & 1 \\ 1 & f_2' \end{vmatrix} / (s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1) \tag{12}$$

which simplifies as

$$\varepsilon_1 = \frac{s_1\bar{\sigma}_2 + s_2}{s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1}. \tag{13}$$

This expression allows a reinterpretation of the Slutsky equation because the income effect is given in (11) and the Slutsky elasticity of substitution is $e_{12} \equiv s_2(s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1)^{-1}$ (Frisch, 1959, p. 180 and line (6) above)[5]. Therefore (13) may be rewritten in its familiar Slutsky form:

$$\varepsilon_1 = s_1\eta_1 + e_{12}. \tag{14}$$

[4] If the reader is willing to let preference elasticities measure the degree of necessity then we may say spending shifts to the less essential.

[5] Note that the Slutsky elasticities of substitution are not symmetric.

The Slutsky equation (14) appears not to allow any further analysis as the income and substitution effects appear to be separate and independent concepts but the Slutsky equation (13) clearly allows simplification and a new interpretation. From (13) demand is elastic if $\bar{\sigma}_1 < 1$ and the good is locally inessential.

An examination of the cross-price elasticity is equally revealing. The preference elasticity approach demonstrates that

(15) $$\varepsilon_{21} = \frac{s_1(1-\bar{\sigma}_1)}{s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1}$$

which may be rewritten in its Slutsky form as

(16) $$\varepsilon_{21} = e_{21} - s_1\eta_2.$$

Once again, the preference elasticity form in (15) admits of further analysis while the traditional Slutsky equation in (16) does not. From (15) we know that $\varepsilon_{21} > 0$ and $x_2$ is a gross substitute for $x_1$ if $\bar{\sigma}_1 < 1$. We have proven the following proposition:[6]

**Proposition 2:** *Given the general conditions then goods may be divided into three categories:*

*Three way taxonomy of goods*

| $\bar{\sigma}_i < 1$ | $\bar{\sigma}_i = 1$ | $\bar{\sigma}_i > 1$ |
|---|---|---|
| *$x_i$ is locally inessential* | *$x_i$ is locally essential* | *$x_i$ is locally essential* |
| $\varepsilon_1 > 1$ | $\varepsilon_1 = 1$ | $\varepsilon_1 < 1$ |
| $\varepsilon_{ji} > 0$ | $\varepsilon_{ji} = 0$ | $\varepsilon_{ji} < 0$ |

6 The closest precursor we know of is Fisher (1972). All his work is conducted within the Slutsky tradition.

Figure 4 illustrates the proposition. The indifference curves are derived from a Cobb-Douglas utility function and therefore illustrate the middle condition where goods are independent.

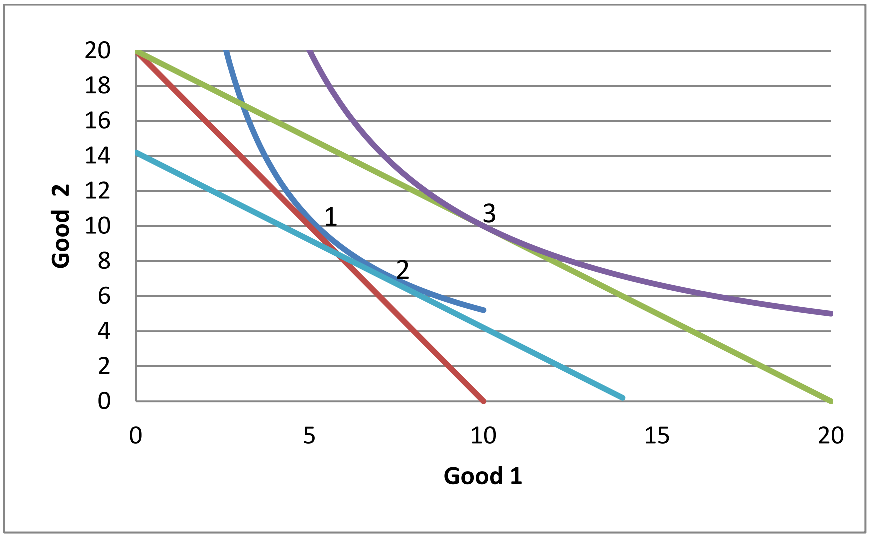


Figure 4: Preference Elasticities and the Slutsky Decomposition

The traditional analysis decomposes the movement from 1 to 3 into the (Hicksian) substitution effect from 1 to 2 and the income effect from 2 to 3.[7] It seems more direct to us to simply discuss the movement from 1 to 3 through $\bar{\sigma}_1$. If the change in the slope of the indifference curve exactly matches the change in the slope of the budget line, as it does for unitary elastic preferences, then there is no change in the demand for $x_2$ and $\bar{\sigma}_1 = 1$. This is the boundary between essential and inessential goods because the budget line and indifference curve become vertical together.[8] Now consider the case where preferences are inelastic,

[7] The Slutsky decomposition depends on an income-compensated substitution effect adding another layer of complexity.

[8] A vertical indifference curve and budget line includes the point where $x_1$ and $x_2$ are both zero and utility is zero. Therefore if $x_1 = 0$ utility is zero independent of the amount of $x_2$ and $x_1$ is essential.

$\bar{\sigma}_1 < 1$. This is the case where indifference curve bend less implying demand for $x_2$ rises as $p_1$ rises and the indifference curves will ultimately cut the axis indicating that $x_1$ is inessential and that $x_2$ is a gross substitute. If preferences are elastic, $\bar{\sigma}_1 > 1$, the indifference curve bends more, $x_2$ is a gross complement and $x_1$ is essential. Non-homothetic utility functions are analyzed in the same way, the only difference is that we need to rotate the budget line toward each axis independently in order to evaluate $\bar{\sigma}_1$ and $\bar{\sigma}_2$ separately.

### *B. Giffen Goods*

For Giffen goods $\varepsilon_1 < 0$ and from (13) it must be true that $s_1\bar{\sigma}_2 + s_2 < 0$. The general conditions assure us that $s_1\bar{\sigma}_2 + s_2\bar{\sigma}_1 > 0$ and it is immediately apparent that $\bar{\sigma}_1 > 1$. Therefore Giffen goods are locally necessary and inferior.

**Proposition 3:** *Given the general conditions above, Giffen goods are locally necessary and inferior.*

## III. Applications

### *A. Marginal Revenue*

The practical implications of the failure to fully simplify the Slutsky decomposition are widespread. One of the more important is that it prevented providing a utility foundation for marginal revenue for over 75 years. That foundation is now easily provided. From (13) marginal revenue may be written as

$$mr = p_1\left(1 - \frac{1}{\varepsilon_1}\right) = p_1\left(\frac{1-\bar{\sigma}_1}{(s_1/s_2)\bar{\sigma}_2 + 1}\right). \tag{17}$$

If $\bar{\sigma}_1 < 1$ then demand is elastic and marginal revenue is positive. As the price of the good is reduced, elastic demand assures us that $s_1 / s_2$ rises as revenue is redirected from $x_2$ to $x_1$. Therefore the lower price and the change in the $s_1 / s_2$ ratio are reinforcing: marginal revenue falls because the price falls and because marginal revenue is a smaller fraction of the price. Marginal revenue may be upward sloping only if one or both of the preference elasticities decline in value indicating that the two goods become closer substitutes at lower prices.

*B. Common Utility Functions*

Another advantage of the preference elasticity approach is that connecting to common utility functions is facilitated because most utility functions are built up from the M.R.S. For example the Cobb-Douglas utility function $U = x_1^{\alpha} x_2^{1-\alpha}$ generates $f = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1}$ and $\bar{\sigma}_2 = \bar{\sigma}_1 = 1$. From here we can simply write down the various elasticities and marginal revenue by consulting their formulas: $\sigma = 1$, $\eta_i = 1$, $\varepsilon_i = 1$, $\varepsilon_{ij} = 0$ and marginal revenue is zero.

For the CES utility function $U = \left(x_1^{\rho} + x_2^{\rho}\right)^{-\rho}$ and $f = \left(x_2/x_1\right)^{1-\rho}$. The preference elasticities are $\bar{\sigma}_2 = \bar{\sigma}_1 = 1-\rho$. Given that utility is homothetic, goods must be normal and $\bar{\sigma}_i > 0$ for $i = 1,\ 2$. Therefore $1 > \rho \geq -\infty$. We may once again write down the various elasticities by consulting the formulas: $\sigma = \frac{1}{1-\rho}$, $\eta_i = 1$, $\varepsilon_i = s_1 + s_2 / (1-\rho)$, $\varepsilon_{ij} = \left(s_1 \rho\right) / \left(1-\rho\right)$ and marginal revenue is $p_1\left(\frac{\rho}{\left(s_1 / s_2\right)(1-\rho)+1}\right)$. For $\rho > 0$ demand is elastic and marginal revenue positive implying that $\left(s_1 / s_2\right)$ rises as $p_1$ declines and marginal revenue is a smaller fraction of a declining price for CES utility. For CES utility then, if marginal revenue is positive it is downward sloping and if marginal revenue is negative it is upward sloping.

Both of these utility functions may be adapted for essential goods by adding a minimum subsistence requirement and this will make the utility functions non-homothetic. Since the adaptation is similar for both, we provide only the Stone-Geary generalization. Let $\gamma_i$ be a shift parameter for the utility function for $x_i$. Typically $\gamma_i$ is a minimum requirement and $\gamma_i > 0$ but it is possible for $\gamma_i$ to represent an endowment of the good and for $\gamma_i < 0$. The Stone-Geary utility function then is $U = \left(x_1 - \gamma_1\right)^{\alpha}\left(x_2 - \gamma_2\right)^{1-\alpha}$ and $f = \frac{\alpha}{1-\alpha}\frac{x_2 - \gamma_2}{x_1 - \gamma_1}$. The constrained elasticities are $\bar{\sigma}_i = \frac{x_i}{x_i - \gamma_i}$ for $i = 1, 2$. These elasticities then are greater than one for necessities and approach infinity as consumption is reduced to the subsistence limit. The elasticity of substitution is

(18) $$\sigma = \frac{1}{s_1\left(\frac{x_2}{x_2 - \gamma_2}\right) + s_2\left(\frac{x_1}{x_1 - \gamma_1}\right)}$$

and at either subsistence boundary $\sigma \to 0$. The elasticity of demand is

(19) $$\varepsilon_1 = \frac{s_1\left(\frac{x_2}{x_2 - \gamma_2}\right) + s_2}{s_1\left(\frac{x_2}{x_2 - \gamma_2}\right) + s_2\left(\frac{x_1}{x_1 - \gamma_1}\right)}$$

so that the demand for these necessities are always inelastic and are completely inelastic at the subsistence limit for $x_1$.

*C. Three or More Goods*

The extension to three or more goods may be outlined quickly. In the comparative static exercise we change the price of one good and hold all other goods prices constant. Therefore we may construct an aggregate of all other goods and the results extend immediately under the Hicks-Leontiev composite commodity theorem (Lewbel, 1995, p. 525).

An alternative approach extends the link between own price elasticity and gross substitution to any number of goods. Differentiate the income constraint $y = p_1 x_1 + \sum_{i=2}^{n} p_i x_i$ with respect to $p_1$ to find that

$$0 = \frac{x_1}{x_1} + \frac{p_1}{x_1}\frac{dx_1}{dp_1} + \sum_{i=2}^{n} \frac{p_i x_i}{p_1 x_1}\frac{p_1}{x_i}\frac{dx_i}{p_1}$$

and rearrange as

(20) $$\varepsilon_1 - 1 = \sum_{i=2}^{n} \frac{s_i}{s_1}\varepsilon_{i1}.$$

We have found that demand is elastic if and only if other goods are on average gross substitutes, with income shares providing the weights for the average. The reason for the result is straightforward. If demand is elastic, then a price increase results in more than proportionate decline in the use of the good releasing revenue for other goods whose demand must rise.

The appendix shows how to extend preference elasticities to three goods. The intuition that $\bar{\sigma}_1 > 1$ is associated with essentials carries through but is modified by the inherent ambiguities once there are more than two goods.

## IV. Discussion

It is not possible to press human subjects to subsistence levels in experiments so that direct evaluation of the forgoing theory is difficult however Robson (2001) argues that utility emerges early in the evolutionary scale suggesting animal studies may be useful. Kagel, Battalio, Rachlin and Green (1981) test the choices of rats in experiments that focus on the distinction between essentials and luxuries. For the experiments on necessities, rats press levers that deliver the only available supplies of food and water. The total number of lever presses is fixed creating a bounded surface much like income and the number of presses for each commodity is varied to alter the slope of the budget constraint much like prices. The experiments for inessentials are similar except that ample supplies of food and water are always available and the inessentials are commodities like cherry cola or root beer. They find that "Essential commodities are determined to be gross complements, while non-essential goods are independent or gross substitutes." (1981, p. 1) This conforms quite well with our tri-partite division of goods where essential goods are gross complements and inessential goods are gross substitutes.[9]

Humans however may perceive luxury and necessity quite differently than animals and attribute a wide range of social values to the distinction. For example Maslow's (1943) hierarchy of needs includes hunger and thirst as well as self-actualization. Therefore human perception of necessity may not fully conform to the behavior of rats. In a study by Kemp (1998) humans are asked to rate goods on a 9 point scale as luxuries or necessities with 9 being a complete luxury and 1 a complete necessity. They are then asked to forecast their relative demand giving a doubling of the price of the good. If demand falls by less than half then

[9] We differ on the assignment of independent goods. We claim independent goods are essentials, they argue these are inessential. A careful reading of their paper though indicates that the conclusion depends on the failure of the data to reject the null hypothesis of independent goods. Therefore their conclusion may represent data limitations and is not necessarily in conflict with our theoretical result.

demand is inelastic, more than half, demand is elastic. If we eliminate the goods given a rating of 4.5 to 5.5 then 12 of 15 goods are rated consistent with our theory: luxuries have elastic demand and necessities inelastic demand.[10] Given the aggregation result in (20) necessities are gross complements and luxuries gross substitutes so that in fact the human and rat studies line up quite well.

None of this should be taken as suggesting that our work is complete or that there are no anomalies left to explain. There are a number of areas where further work is necessary and may produce contrary results. For example necessities come in many varieties and given the presence of one variety the others may be inessential. Different sources of the same amino acid, different flavors of water may well be close substitutes for each other even as the group as a whole is essential. We believe that some form of aggregation into goods that serve particular needs will ultimately prove fruitful. We suspect that the core idea is simply that if either of two goods may be driven to zero but not both then the goods are different varieties of the same essential aggregate. However, we have no formal results to offer in support of this belief. Similarly two non-essential goods may be complementary such as each earring in a pair. These too we would aggregate as a single non-essential good: one pair of earrings.

We remain optimistic that our approach will prove fruitful in part given the difficulties the approach based on substitutes and complements has encountered. For an entertaining survey of these problems the reader can do no better than read Samuelson's (1974) classic treatment. The fact the broad outlines of our approach have been confirmed by human and animal studies is also reassuring.

[10] The exceptions are chocolate, rated as a luxury with inelastic demand and bus trips and milk both rated as essentials with elastic demand. For the six goods eliminated from the analysis, 3 are consistent and 3 inconsistent with our theory.

## V. Conclusion

Preference elasticities are the keys that link substitutes, complements, needs, wants, demand elasticity, income effects and marginal revenue together. Elasticities greater than one imply the marginal rate of substitution is changing rapidly enough that the indifference curve will not cross the axis and utility is undefined unless the good is consumed. This is the natural definition of a necessity. The same rapid change in the slope of the indifference curve indicates that a higher price of the good will draw income from the other good and reduce the quantity demanded, the definition of a gross complement. That necessities should be complementary follows naturally from the concept that each is essential for positive utility.

We also have a solution to one of the longest standing and fundamental puzzles in economics: why can A be a gross substitute for B while B is a gross complement of A? The numeric answer that B has a higher income affect has long been known. To this we may now add that A is a need and B is a want which provides a natural explanation. Adding A satisfies a hunger causing the MRS to change rapidly and as income rises, spending shifts rapidly to B. A lower price for either good causes a small substitution effect and a large income effect toward B. A new type of symmetry emerges: the response to either price decline is essentially the same, the quantity demanded of B rises as the dominant effect is that income shifts away from the essential.

The core ideas extend to any number of goods in a straightforward way. Necessities have inelastic demand because the price must rise more than in proportion to the quantity decline in order for the demand curve to be bounded away from the axis. Therefore revenue is attracted from other goods and necessities are on average gross complements with other goods. Given two goods, higher preference elasticities produce low elasticities of substitution and, as the appendix shows, adding a necessity reduces elasticities of substitution in larger systems.

That the elasticity of the MRS should play a significant role in demand analysis is so natural and obvious that the real question is why it has lingered so long in the shadows. This question we cannot answer. We only know that it is our good fortune to discover that an approach based on needs and wants spotlights this important concept.

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**Appendix – Generalizing preference elasticities to many goods.**

We are interested in the utility foundation for needs and wants where demand crosses the vertical axis for wants but not needs. If the elasticity of demand is 1, then the demand curve is a rectangular hyperbola that approaches but never crosses the vertical axis. This divides needs from wants. If demand elasticity is less than one then the price increase is faster and the asymptotic approach to the axis is slower again implying the good is essential. If demand touches the axis, the elasticity is greater than one. The proof is simple: if demand touches the axis revenue falls from something positive to zero and declining revenue with a rising price is the hallmark of elastic demand. Therefore we are interested in the underlying utility structure that separates elastic from inelastic demand.

We begin with the first order conditions for three goods:

(A1)
$$\frac{y}{p_2} = \frac{p_1}{p_2}x_1 + x_2 + \frac{p_3}{p_2}x_3$$
$$f = \frac{p_1}{p_2}$$
$$g = \frac{p_3}{p_2}$$

where $g \equiv u_3'/u_2'$. Differentiating with respect to all three inputs, income and the price of good one we have

(A2)
$$\begin{bmatrix} f & 1 & g \\ f_1' & f_2' & f_3' \\ g_1' & g_2' & g_3' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \frac{1}{p_2} \begin{bmatrix} dy - x_1 dp_1 \\ dp_1 \\ 0 \end{bmatrix}.$$

Now the ambiguity associated with three goods is clear: in general we will not know the signs of $f_3'$ or $g_1'$ as each refers to the change in the MRS between two goods with respect to a third good.

Once again, the elasticity of substitution is closely related to the determinant of the system and is a weighted average of preference elasticities. Let $M$ be the square matrix above and let $M_i$ represent the cofactor of element $i$ in the first row of $M$. For well behaved utility functions the expected sign of the determinant for $n$ goods alternate in sign as $n$ increases. In our case, systems with an even number of goods have a positive sign, odd numbers of goods a negative sign.

The preference elasticities given three goods then are defined as

(A3) $$\bar{\sigma}_{jk} = -\frac{x_j x_k}{fg} M_i$$

where $i \neq j \neq k$.[11] These elasticities hold one good constant: the cofactors of the first row eliminate all the derivatives with respect to $x_i$. This allows us to see how a single good affects the overall elasticity of substitution. Following Hicks and Allen (1934) the elasticity of substitution is

(A4) $$\sigma(f,g) = -\left[\frac{p_2 x_1 x_2 x_3}{yfg}\Delta\right]^{-1} = \frac{1}{s_1\bar{\sigma}_{23} + s_2\bar{\sigma}_{13} + s_3\bar{\sigma}_{12}}.$$

The notation $\sigma(f,g)$ will allow us to discuss how the elasticity of substitution with three goods is related to the elasticity of substitution of either two good system: $\sigma(f)$ or $\sigma(g)$ which will prove to be valuable.

The elasticity of demand is

[11] For two goods $\bar{\sigma}_j = \frac{x_j}{f} M_i \ (i \neq j)$.

(A5)
$$-\frac{p_1}{x_1}\frac{dx_1}{dp_1} = \frac{p_1}{x_1}\frac{p_2}{y}\frac{x_1x_2x_3}{fg}\frac{1}{p_2}\begin{vmatrix} -x_1 & 1 & g \\ 1 & f_2' & f_3' \\ 0 & g_2' & g_3' \end{vmatrix}\sigma(f,g)$$
$$= \frac{s_1\bar{\sigma}_{23}(f,g) + s_2\bar{\sigma}_3(g) + s_3\bar{\sigma}_2(g)}{s_1\bar{\sigma}_{23}(f,g) + s_2\bar{\sigma}_{13}(f,g) + s_3\bar{\sigma}_{12}(f,g)}.$$

In general it is difficult to compare elasticities of substitution for systems with different numbers of goods but the expression above solves the problem quite nicely. Notice that whether demand is elastic or not depends entirely on whether the elasticity of substitution between the two goods in $\sigma(g)$ rises or falls with the addition of $x_1$. If adding the good provides some relief from diminishing returns, as measured by $\bar{\sigma}_{1j}(f,g) < \bar{\sigma}_j(g)$ for $j = 2,3$ then the demand for the good is elastic and the good is a want. On the other hand, if diminishing returns intensify, the good is a need.

We may tie this result to the results with two goods by expanding out the expression $\bar{\sigma}_{1j}(f,g)$. For example,

(A6)
$$\bar{\sigma}_{13}(f,g) = \bar{\sigma}_1(f)\bar{\sigma}_3(g) - \bar{\sigma}_3(f)\bar{\sigma}_1(g).$$

The first two terms are the familiar preference elasticities for two goods while the last two are the elasticities of MRS with respect to a third good. In general these cannot be signed and if we assume $\bar{\sigma}_1(g) = \bar{\sigma}_3(f) = 0$ then $\bar{\sigma}_{13}(f,g) > \bar{\sigma}_3(g)$ if and only if $\bar{\sigma}_1(f) > 1$. Under these conditions, just as in the section on two goods, demand is inelastic and the good is a need if $\bar{\sigma}_1(f) > 1$. Therefore the rule we learned for two goods extends directly to three or more goods with adjustments for the dependence of the MRS between $i$ and $j$ on $x_k$.